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THE PROCESS OF BUILDING UP THE VOLTAGE AND CURRENT IN A LONG ALTERNATING-CURRENT CIRCUIT.

By A. E. Kennelly.

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Let us consider the simple alternating-current circuit indicated in Figure 1; namely, a long single uniform wire AB, with a ground return circuit. A single-phase sinusoidal alternator of negligible internal impedance and generating an e. m. f.

$$E = E_A \sin \omega t \qquad \text{abvolts} \qquad (1)$$

is connected to the end A at the instant $t = 0$, when E is starting positively from zero at the uniform frequency n cycles per second, or with angular velocity $\omega = 2\pi n$ radians per second. At B, the distant end of the line, there is connected an impedance Z to ground. If we make $Z = \infty$, we virtually free or disconnect the line at B. If we make $Z = 0$, we virtually ground the line at B. If we make $Z = z_r$, we virtually insert a receiving impedance at B. Any two-wire or multiple-wire metallic circuit is capable of being reduced to an equivalent simple single-wire circuit with ground return, such as is represented in Figure 1.

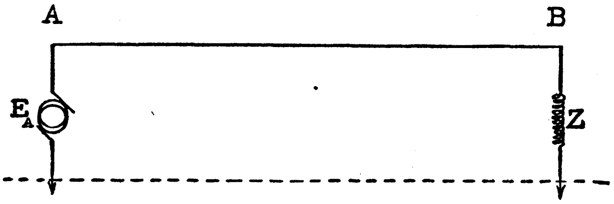

Figure 1. — Simple alternating-current circuit.

The current and voltage which exist at any point of the line in Figure 1 can be readily computed, whatever the given terminal impedance at B, after the system has been allowed to operate for a sufficient length of time, usually about one second, during which the initial unsteady state may be regarded from a practical standpoint as developing into the final steady state. It is proposed to discuss in this paper certain formulas, that appear to be new in this application, by which the actions occurring during the initial unsteady state can be easily grasped and followed to the final steady state.[1] It will be readily seen by the aid of these formulas that the final steady state is approached by successive sudden jumps, and not by a uniform continuous advance. The unsteady state is a state of leaps and bounds; but these, although not necessarily or generally of successively diminishing magnitude, diminish in a definitely irregular way, and finally disappear. From a physical point of view, the process is very beautiful, and from a practical engineering standpoint, it is not without importance.

We may first derive the formulas for voltage and current in the circuit of Figure 1 in the usual way, starting with the differential equations of the circuit, and then lead to these already known formulas in the new way.

At any instant t, and at any distance x centimeters from A along the line in Figure 1, we have[2]

$-e' = zi$ abvolts per cm. (2)

and $-i' = ye$ absamperes per cm. (3)

where $e =$ the instantaneous e. m. f., abvolts

$e' = \frac{de}{dx}$ the space gradient of e $\frac{\text{abvolts}}{\text{cm.}}$

$i =$ the instantaneous current, absamperes

$i' = \frac{di}{dx}$ the space gradient of i $\frac{\text{absamperes}}{\text{cm.}}$

$r =$ conductor linear resistance, $\frac{\text{absohms}}{\text{cm.}}$

$l =$ conductor linear inductance, $\frac{\text{abhenrys}}{\text{cm.}}$

$g =$ dielectric linear conductance, $\frac{\text{abmhos}}{\text{cm.}}$

$c =$ dielectric linear capacity $\frac{\text{abfarads}}{\text{cm.}}$

$z = r + l\frac{d}{dt}$ $\frac{\text{absohms}}{\text{cm.}}$ (4)

$y = g + c\frac{d}{dt}$ $\frac{\text{abmhos}}{\text{cm.}}$ (5)

$\frac{d}{dt} =$ time-rate of change $\frac{1}{\text{second}}$

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[1] Compare, however, "Alternating Currents," by Bedell and Crehore, 1893, pp. 201-207.

[2] "Electromagnetic Theory," by Oliver Heaviside, **1**, 450 (1893).

Equation (2) states that the descending gradient of voltage in abvolts per linear centimeter at any point of the line is always equal to the current strength at that point multiplied by the conductor resistance per centimeter plus the time-rate of change of the current into the conductor inductance per centimeter. In other words, the voltage per linear centimeter is equal to the momentary linear Ir drop, plus the linear back e. m. f. of self-induction.

Equation (3) states that the descending gradient of current along the line is always equal to the local voltage multiplied by the linear dielectric conductance, plus the linear capacity into the local time-rate of change of the voltage, i. e. the linear leakage current plus the linear charging current.

Differentiating (2) and (3) with respect to x the length of line, we get

$$e'' = -zi' = yz \cdot e \qquad \frac{\text{abvolts per cm.}}{\text{cm.}} \qquad (6)$$

$$i'' = -ye' = yz \cdot i \qquad \frac{\text{absamperes per cm.}}{\text{cm.}} \qquad (7)$$

By (6) the curvature (or gradient of gradient) of the voltage at any point along the line is always yz times the voltage at that point. By (7) the curvature of the current along the line is also yz times the local current. The complete solutions of the second-order differential equations (6) and (7), for any point distant x from A, are known to be :

$$e = E \cosh x\sqrt{yz} - I\sqrt{\frac{z}{y}} \sinh x\sqrt{yz} \qquad \text{abvolts} \qquad (8)$$

$$i = I \cosh x\sqrt{yz} - E\sqrt{\frac{y}{z}} \sinh x\sqrt{yz} \qquad \text{absamperes} \qquad (9)$$

where E and I are the instantaneous impressed voltage, and current, at the sending end A (Figure 1), respectively.

For an impressed e. m. f. which is sinusoidal, or simply harmonic, in accordance with (1), we have, if $j = \sqrt{-1}$,

$$\frac{dE}{dt} = E_A \omega \cos \omega t = j\omega E \qquad \text{abvolts per second} \qquad (10)$$

and

$$\frac{dI}{dt} = I_A \omega \cos \omega t = j\omega I \qquad \text{absamperes per second} \qquad (11)$$

so that

$$z = r + jl\omega \qquad \text{absohms per cm.} \qquad (12)$$

$$y = g + jc\omega \qquad \text{abmhos per cm.} \qquad (13)$$

y and z are thus plane-vector constants.

We have hitherto kept the equations within the absolute C. G. S. magnetic system of units. We may, however, transfer them to the "practical" system and use the kilometer (or the mile) as the unit of length :[3]

$$e = E \cosh L_1 \alpha - I z_0 \sinh L_1 \alpha \qquad \text{volts} \qquad (14)$$

$$i = I \cosh L_1 \alpha - \frac{E}{z_0} \sinh L_1 \alpha \qquad \text{amperes} \qquad (15)$$

where L_1 is the distance x measured from A in kilometers,

$$\alpha = \sqrt{(r + jl\omega)\,(g + jc\omega)} \qquad \frac{1}{\text{kilometer}} \qquad (16)$$

and

$$z_0 = \sqrt{\frac{r + jl\omega}{g + jc\omega}} \qquad \text{ohms} \qquad (17)$$

the vector, or complex quantity, α is commonly called the attenuation-constant of the circuit. It comprises a "real" part α_1 called the real attenuation-constant, and an "imaginary" part $j\alpha_2$ called the wave-length constant, by the relation

$$\alpha = \alpha_1 + j\alpha_2 \qquad \frac{1}{\text{kilometer}} \qquad (18)$$

The vector impedance z_0 is called the "initial sending-end impedance" of the line, and sometimes the "surge-impedance."

r = the linear resistance . . . . . ohms per kilometer.
l = the linear inductance . . . . . henrys per kilometer.
g = the linear leakage conductance . . mhos per kilometer.
c = the linear capacity . . . . . farads per kilometer.

Since equations (14) and (15) hold for any moment of time, they must apply to the final steady state. We may easily derive from them the following steady-state formulas :

1. When the distant end of the line at B is free,

$$I_B = 0 \qquad I_A = \frac{E}{z_0 \coth L\alpha} \qquad \text{amperes} \qquad (19)$$

$$E_A = E \qquad E_B = \frac{E}{\cosh L\alpha} \qquad \text{volts} \qquad (20)$$

2. When the distant end of the line at B is grounded,

$$I_B = \frac{E}{z_0 \sinh L\alpha} \qquad I_A = \frac{E}{z_0 \tanh L\alpha} \qquad \text{amperes} \qquad (21)$$

$$E_A = E \qquad E_B = 0 \qquad \text{volts} \qquad (22)$$

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[3] "The Distribution of Pressure and Current over Alternating-Current Circuits," by A. E. Kennelly. Harvard Engineering Journal, 1905–1906.

3. When the distant end of the line at B is grounded through a receiving impedance z_r ohms,

$$I_B = \frac{E}{z_0 \sinh La + z_r \cosh La} \qquad \text{amperes} \qquad (23)$$

$$E_B = I_B z_r \qquad \text{volts} \qquad (24)$$

All of the values of voltage and current in the steady-state formulas (14) to (24) inclusive, may be regarded either as instantaneous voltages and currents, in terms of instantaneous impressed voltage E and current I; or they may be regarded as effective, or square-root-of-mean-square values, in terms of effective impressed voltage E and current I, such as would be indicated by properly calibrated voltmeters and ammeters.

We may now proceed to show how formulas (19) to (24) inclusive may be derived by taking into account the initial outgoing waves of voltage and current, together with superposed reflected waves. The only postulate needed is that when the line is first connected to the alternator at A, the first outgoing current wave has the strength

$$I_0 = \frac{E}{z_0} \qquad \text{amperes} \qquad (25)$$

and this current strength continues to be delivered to the line at A until such time as current-waves reflected from the distant end modify the current. This proposition is well known.[4]

The outgoing voltage and current waves run along the wire hand in hand, with the velocity:

$$v = \frac{\omega}{a_2} \qquad \text{kilometers per second} \qquad (26)$$

a velocity which is somewhat less than the speed of long-wave light in the dielectric. When the linear conductor resistance and leakage conductance are very small, this velocity approximates to:

$$v = \frac{1}{\sqrt{cl}} \qquad \text{kilometers per second} \qquad (27)$$

In the case of overhead aerial copper wires, v approaches 300,000 kilometers per second at high frequencies. At low frequencies the effect of conductor resistance requires the use of formula (26), and may bring down the velocity to 100,000 kilometers per second or less. In paper-insulated, and especially in rubber-insulated cables, the velocity by (26) may be reduced, with low frequencies, to 10,000 kilometers per second, or less.

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[4] "On the Mechanism of Electric Power Transmission," by A. E. Kennelly. Electrical World, **42,** 673 (Oct. 24, 1903).

As the electromagnetic wave starting from A at time $t = 0$ runs over the circuit, containing within it both a voltage wave, or electrostatic-flux wave, and a current wave, or magnetic-flux wave, it dwindles or attenuates at the rate of $\epsilon^{-\alpha}$ per kilometer; so that after running L_1 miles, both the voltage and the current will have become weakened by the attenuation-coefficient $\epsilon^{-L_1\alpha}$; or

$$e = E\epsilon^{-L_1\alpha} \quad \text{volts} \quad (28)$$

and

$$i = I_0\,\epsilon^{-L_1\alpha} \quad \text{amperes} \quad (29)$$

where α is the attenuation-constant of formula (16). This logarithmic attenuation is represented for a particular circuit and frequency in Figure 2, but the same curve, A B C D, may be used to represent any circuit and frequency, by suitably changing the scales of co-ordinates. The length of the horizontal axis is determined by α_1, the real part of the attenuation-constant α in (18), and the wave-length is determined by α_2, the imaginary part. The greater α_1, the shorter the length of circuit that will fit the curve A B C D.

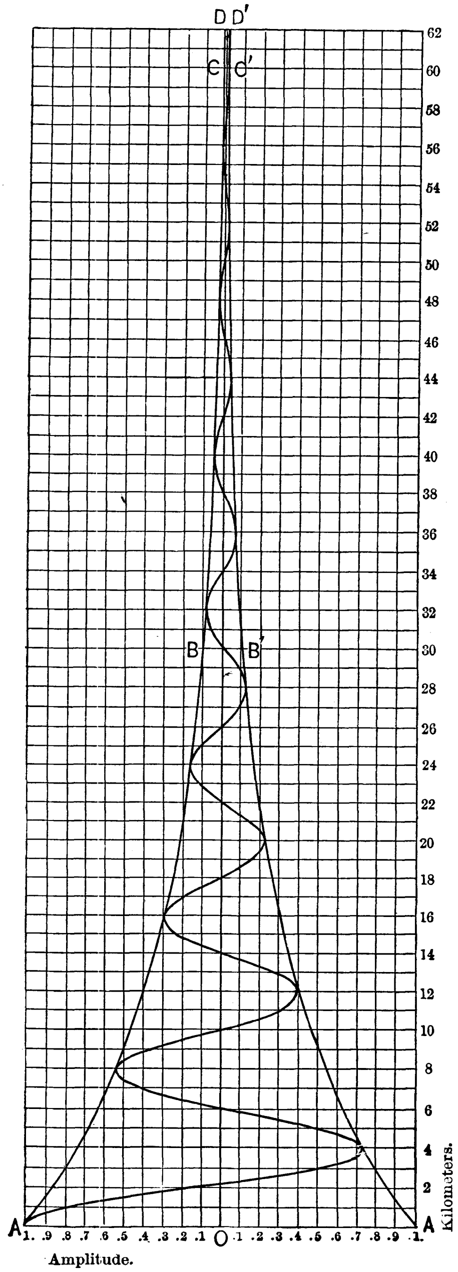

FIGURE 2. — Curve of wave attenuation for the first 62 kilometers of a circuit of attenuation-constant $\alpha = 0.07675 + j0.7854$ per kilometer.

The greater α_2, the shorter the wave-length, or the more waves that can be filled into the diagram.

In the case represented by Figure 2, $\alpha = \alpha_1 + j\alpha_2 = 0.07675 + j\,0.7854$ per kilometer. After running, say thirty kilometers, the wave will have dwindled by the attenuation-coefficient $\epsilon^{-30\,(0.07675 + j\,0.7854)} = \epsilon^{-(2.3026 + j\,23.562)} = \epsilon^{-2.3026} \times \epsilon^{-j\,23.562}$. The first factor is equal to numeric 0.1. The second is equal to a negative angle of 23.562 radians, or $3\frac{3}{4}$ complete rotations or cycles. The waves of current and voltage on arriving at B have thus each dwindled to 10 per cent of their original amplitude, and have also fallen $3\frac{3}{4}$ cycles behind the phase existing at A. The phase of the wave remains the same as it advances, a crest remaining a crest, but the phase at A is constantly advancing at the angular velocity ω radians per second, so that with respect to the phase at A, that of the advancing wave is constantly falling behind.

After advancing another thirty kilometers, or to C, Figure 2, sixty kilometers from A, the voltage and current waves will again shrink to 10 per cent of their amplitudes at B, or to 1 per cent of their original amplitudes, and will have fallen $7\frac{1}{2}$ cycles in phase behind the voltage and current phases at A, respectively, the attenuation-coefficient having become $\epsilon^{-(4.6052 + j47.124)} = 0.01 \diagdown\!\overline{2700^\circ}$. The original phase difference between the voltage and current waves at A is equal to the angle of the vector z_0, the initial sending-end impedance (17). With negligible resistance and leakage conductance, formula (17) shows that the angle of z_0 would approach zero; so that on such a circuit the voltage and current would be in phase at the start, and also at any point along the line. Whatever the original phase difference might be, according to (17), it would be maintained throughout the first run along the line.

If we represent at O E, Figure 3, a unit vector rotating in a vertical plane about a horizontal axis O, with the angular velocity ω radians per second, its projection on the horizontal plane 6 O 2 will at any instant represent the actual voltage applied to the end A of the line. At a distance of one kilometer from A, the phase of the outgoing wave stream will be O 1, or 45° behind O E. The amplitude will have shrunk to O 1′ or 0.923, an attenuation of 7.7 per cent. The projection of O 1′ on the horizontal plane 6 O 2 will give the amplitude of the wave at the same instant. At a distance of another kilometer beyond, or two kilometers from the origin, the phase of the wave stream will be 90° behind O E, and the amplitude will have again lost 7.7 per cent, or will have fallen to 0.853, 85.3 per cent of the original. By continuing the examination, we should find that at the instant considered, all points along the line would have a wave amplitude determined by the orthogonal projection of a certain radius corresponding thereto in a logarithmic spiral E 1′ 2′ 3′, etc. At successive distances of eight kilometers, the phases at any instant would be the same. Moreover, by rotating the entire spiral system of Figure 3 counter-clockwise about O, with angular velocity ω, the amplitude and phase of the wave stream at any point on the line could be determined, for any moment, by the orthogonal instantaneous projection of the proper radius in the spiral.

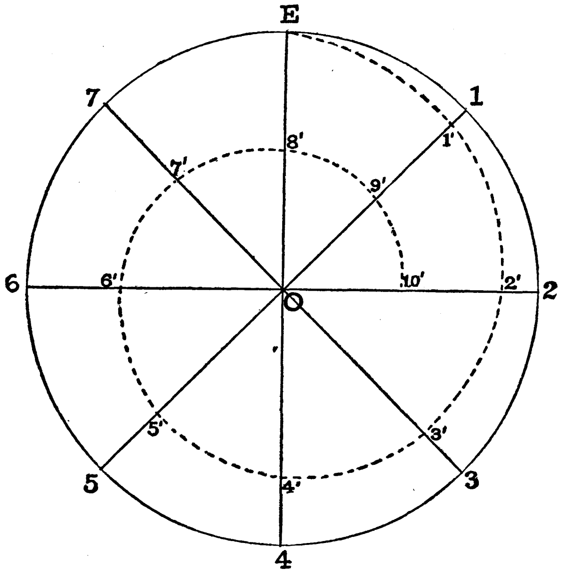

FIGURE 3. — Diagram of relative magnitudes and phases of outgoing wave over the line of Figure 2, for the first ten kilometers.

By combining Figures 2 and 3, a mechanical model might be prepared to illustrate the first run of an electromagnetic wave over a circuit. Into a cylindrical wooden shaft, of the length O O′, Figure 2, metal pins or wire nails are driven at intervals, corresponding, say, to each quarter kilometer. The radial length of the pin from the axis of the cylinder must conform to the logarithmic curve A B C D, Figure 2, and its angular position about the axis must conform to Figure 3, for the particular line considered, or thirty-two pins to 360°. A shutter S S, Figure 4, movable in a plane parallel to the shaft, like the sliding lid of a box, should start from A and admit vertically-falling light rays L L, on to the spiral, at a moment when the shaft is rotated by the handle H at uniform angular velocity ω, and when the first pin at A stands vertical. The shutter must be geared with the shaft O O′, so as to move uniformly and uncover one complete turn of the spiral in the interval of one complete rotation of the shaft. The shadow of any pin P on a horizontal plane at *p* beneath would then indicate the wave motion at that point from the moment of the wave's first reaching it until reflected waves disturbed the distribution. The spiral of pins thus constructed would just fit into the arrangement of Figure 2, considered as a hollow cylinder or logarithmic trumpet of revolution.

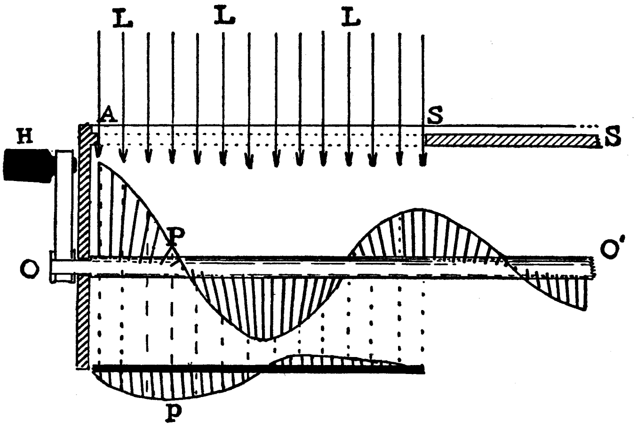

FIGURE 4. — Diagram of model for exhibiting the propagation of the first outgoing wave-train along a uniform line.

In the case of an indefinitely long circuit, as above outlined, the initial current strength at the sending end A would remain the final steady current strength. In other words, there would be nothing to disturb the original simple harmonic voltage and current. At the origin A we should always have:

$$E = E_A \sin \omega t \qquad \text{instantaneous volts} \qquad (30)$$

or

$$\overline{E} = \frac{E}{\sqrt{2}} \qquad \text{effective volts} \qquad (31)$$

also

$$I_0 = I_{0A} \sin \omega t = \frac{E_A}{z_0} \sin \omega t \qquad \text{instantaneous amperes} \qquad (32)$$

or

$$\overline{I}_0 = \frac{I_0}{\sqrt{2}} \qquad \text{effective amperes} \qquad (33)$$

At any distance L_1 kilometers from A we should have

$$e = E\,\epsilon^{-L_1\alpha_1}\,\overline{\diagdown\alpha_2} \quad = E\,\epsilon^{-L_1\alpha} \qquad \text{instantaneous volts} \qquad (34)$$

or

$$\bar{e} = \frac{E}{\sqrt{2}}\,\epsilon^{-L_1\alpha_1}\,\overline{\diagdown\alpha_2} \quad = \frac{E}{\sqrt{2}}\,\epsilon^{-L_1\alpha} \qquad \text{effective volts} \qquad (35)$$

also

$$i = I_0\,\epsilon^{-L_1\alpha_1}\,\overline{\diagdown\alpha_2} \quad = I_0\,\epsilon^{-L_1\alpha} \qquad \text{instantaneous amperes} \qquad (36)$$

$$\bar{i} = \frac{I_0}{\sqrt{2}}\,\epsilon^{-L_1\alpha_1}\,\overline{\diagdown\alpha_2} \quad = \frac{I_0}{\sqrt{2}}\,\epsilon^{-L_1\alpha} \qquad \text{effective amperes} \qquad (37)$$

If, however, the circuit be comparatively short, and be free, or opened, at the distant end ($Z = \infty$ in Figure 1), the wave stream on arriving at B will be reflected back towards A. Let the length of the line be L kilometers, then the voltage when it reaches B will have the value $E\epsilon^{-L\alpha}$ and the current likewise $I_0\,\epsilon^{-L\alpha}$. The effects of reflection are different on voltage and on current, i. e., on moving electrostatic flux and moving magnetic flux, so that each must be considered separately.

Considering first the voltage wave arriving at the distant open end of the line, it rebounds therefrom at light-speed, or is reflected there. The voltage in the reflected wave at B starts back at the same amplitude and phase as when it strikes the free end. It is simply started backwards along the circuit without undergoing any other change. It returns towards A, attenuating as it goes, just as though nothing had happened. At B, however, after the reflection, the voltage is double that which occurred just before reflection; for as soon as the reflected wave starts back, there are the $E\epsilon^{-L\alpha}$ volts arriving at the head of the incoming train, and also $E\epsilon^{-L\alpha}$ volts starting at the head of the returning train. The voltage at B was thus 0 until the wave arrived at B, then it was $E\epsilon^{-L\alpha}$, whereupon reflection immediately occurred and made the voltage jump to $2\,E\epsilon^{-L\alpha}$; where α is a complex number, the real part of $2\,E\epsilon^{-L\alpha}$, being the maximum cyclic value of the voltage at B, and $2\,E\epsilon^{-L\alpha_1}\sin(\omega t - \alpha_2)$ being the voltage at any instant after arrival.

By the time that the wave train gets back to A, the total distance it will have travelled is $2\,L$ kilometers, and the attenuation-coefficient will be $\epsilon^{-2L\alpha}$. The arriving voltage will therefore be $E\epsilon^{-2L\alpha}$. As soon as the end A of the line is reached, the wave will go direct to ground through the alternator. The e. m. f. of the alternator does not stop the wave, which goes straight through the machine (assumed as impedanceless) without modification. A voltage wave passing through a ground or short-circuit virtually swings round the goal and returns on its path reversed. That is the same, however, as being reflected with change of sign, or with a change of 180 degrees, or π radians, in phase. The wave after reaching A and going to ground returns back on the circuit towards B, but reversed in sign. The voltage at A is thus not affected by the return of the wave, because although it tends to be increased by $E\epsilon^{-2La}$ volts on the first return it is immediately increased by $-E\epsilon^{-2La}$ volts on the recoil from A, these two impulses being equal and opposite.

The wave reflected with inversion at A now runs back towards B for the second time. When it reaches B it will have run a total distance of 3 L kilometers from the start, and its magnitude will be $-E\epsilon^{-3La}$ volts. At the open end B, it is reflected without reversal, thus adding a sudden jump of $-2E\epsilon^{-3La}$ to the voltage previously existing at B. Back it comes to A in the condition $-E\epsilon^{-4La}$. It is again reflected from A and ground with reversal, making no change in the voltage impressed on A, but starting out for B with the value $E\epsilon^{-4La}$. It reaches B in the condition $E\epsilon^{-5La}$, adding a sudden jump of $2E\epsilon^{-5La}$ to the voltage existing at that moment, and again sets out for A.

The time of one passage along the line is $\frac{L}{v}$ seconds, where L is the length of the line and v is approximately $\frac{1}{\sqrt{lc}}$ kilometers per second. In overhead aerial wires v is nearly 300,000 kilometers per second or the light speed in air; so that over a circuit 300 kilometers long, the time of flight of a wave would be about $\frac{1}{1000}$ second, and the time of one return trip $\frac{2}{1000}$ second. In one second of time after applying the alternator to A there would have been completed nearly 500 return trips over the circuit.

From time $t = 0$ to $t = 0.001$ second, the voltage at B $= 0$
From time $t = 0.001$ to 0.003 second, the voltage at B $= 2E\epsilon^{-La}$ volts
From time $t = 0.003$ to 0.005 second, the voltage at B $=$
$$2E\epsilon^{-La} - 2E\epsilon^{-3La}$$
From time $t = 0.005$ to 0.007 second, the voltage at B $=$
$$2E\epsilon^{-La} - 2E\epsilon^{-3La} + 2E\epsilon^{-5La}$$

and so on indefinitely, until the succeeding jumps of voltage at B become infinitely small, or the wave has been attenuated away to nothing. The total final voltage at B after the steady state is finally reached is:

$$E_B = 2E\{\epsilon^{-La} - \epsilon^{-3La} + \epsilon^{-5La} - \epsilon^{-7La} + \dots\dots \quad \text{volts} \quad (38)$$
$$= 2E\epsilon^{-La}\{1 - \epsilon^{-2La} + \epsilon^{-4La} - \epsilon^{-6La} + \dots\dots \quad \text{"} \quad (39)$$
$$= 2E\frac{\epsilon^{-La}}{1 + \epsilon^{-2La}} = \frac{2E}{\epsilon^{La} + \epsilon^{-La}} = \frac{E}{\cosh La} \quad \text{"} \quad (40)$$

The final result (40) is the same as was found in formula (20). All the terms in equations (38) to (40) are vector quantities, and are supposed to be added vectorially. It is evident that the e. m. f. impressed at A is unaffected by reflected waves; while the e. m. f. at the distant free end increases by jumps that are numerically smaller and smaller; but the phase relation may be such that the effect on the resultant of any one jump may exceed that of the preceding jump.

In particular cases when the length L of the line is suitably chosen with respect to the frequency and to the line constants, the final voltage at the distant end may greatly exceed the voltage at the sending end, either on the basis of maximum cyclic values, or of effective, i. e. voltmeter, values.

Turning now to the current wave, or magnetic flux wave, when this reaches B for the first time its value is $I_0 \epsilon^{-L\alpha}$, a vector quantity, having a certain phase lag $L\alpha_2$ with respect to the current entering simultaneously at A, and also undergoing simple harmonic variation. When a current wave strikes a free end or discontinuity, it is reflected with inversion; i. e., with reversal of sign, or 180° (π radians), change of phase. It starts back, therefore, towards A as $-I_0 \epsilon^{-L\alpha}$ amperes, and the total current increment at B is nil. On arriving at A its condition is $-I_0 \epsilon^{-2L\alpha}$ amperes. It then passes through the impressed e. m. f. without change and is reflected from the ground. A current reflected from a ground or short circuit undergoes no change of phase. It adds, therefore, a jump of $-2 I_0 \epsilon^{-2L\alpha}$ amperes at A vectorially to the current I_0 impressed there. Whether this will involve an increase or decrease in the magnitude of the current at A will depend upon the relative phases of I_0 and of $-2I_0 \epsilon^{-2L\alpha}$, which in turn will depend on the length of the line L for any given frequency. The current wave then returns to B for the second time, reaching it in the condition $-I_0 \epsilon^{-3L\alpha}$. It is again reflected with inversion or starts back to A as $I_0 \epsilon^{-3L\alpha}$ amperes, adding nothing to the current at B. It keeps up this battledore and shuttlecock action theoretically forever; but practically until it becomes attenuated to insignificant dimensions. The final resulting current at B is zero, which, of course, must be true at an open end. This is due to the occurrence there of pairs of incident and reflected current waves having equal magnitudes but respectively opposite signs.

The total current strength at A is

$$\begin{array}{lll} I_0 & \text{from } t = 0 & \text{to } t = 0.002 \\ I_0 - 2 I_0 \epsilon^{-2L\alpha} & \text{"} \quad t = 0.002 & \text{to } t = 0.004 \\ I_0 - 2 I_0 \epsilon^{-2L\alpha} + 2 I_0 \epsilon^{-4L\alpha} & \text{"} \quad t = 0.004 & \text{to } t = 0.006 \end{array}$$

and so on.

After an indefinitely great lapse of time, the current at A is

$$I_A = I_0 - 2\,I_0\,\epsilon^{-2La} + 2\,I_0\,\epsilon^{-4La} - 2\,I_0\,\epsilon^{-6La} + \,.\,.\,.\,.\,.\quad \text{amperes} \tag{41}$$

$$= I_0\{1 - 2\,\epsilon^{-2La} + 2\,\epsilon^{-4La} - 2\,\epsilon^{-6La} + \,.\,.\,.\,.\,.\quad \text{``} \tag{42}$$

$$= I_0\{1 - 2\,\epsilon^{-2La}\,(1 - \epsilon^{-2La} + \epsilon^{-4La} - \epsilon^{-6La} + \,.\,.\,.\,.\,.\quad \text{``} \tag{43}$$

$$= I_0\left\{1 - \frac{2\,\epsilon^{-2La}}{1 + \epsilon^{-2La}}\right\} = I_0\left\{1 - \frac{2}{\epsilon^{2La} + 1}\right\} = I_0\left\{\frac{\epsilon^{La} - \epsilon^{-La}}{\epsilon^{La} + \epsilon^{-La}}\right\} \tag{44}$$

$$= \frac{I_0}{\coth La} = \frac{E}{z_0 \coth La} \quad \text{amperes} \tag{45}$$

The final result (45) thus agrees with the result of (19).

If now the line be grounded directly at B, the final voltage at B is nil, because each voltage wave is now reflected from that end with inversion. The voltage at A also remains steady at E because the voltage waves are reflected from the ground at A with inversion.

The current arriving at B the first time is $I_0\,\epsilon^{-La}$. It is reflected directly from the ground there, thus producing at B a total increment of $2\,I_0\,\epsilon^{-La}$ amperes. On arriving back at A its value is $I_0\,\epsilon_{-2L}{}^{a}$. Being reflected from the ground at A without inversion, it adds a total increment of $2\,I_0\,\epsilon^{-2La}$ to I_0 the vector current at A. When it reaches B the second time, its value is $I_0\,\epsilon^{-3La}$, and it adds there an increment of $2\,I_0\,\epsilon^{-3La}$. Reversal does not recur at either end of the line. The final current strength at B after a sufficient lapse of time is

$$I_B = 2\,I_0\,\epsilon^{-La} + 2\,I_0\,\epsilon^{-3La} + 2\,I\,\epsilon^{-5La} + \,.\,.\,.\,.\,.\quad \text{amperes} \tag{46}$$

$$= 2\,I_0\,\epsilon^{-La}\,(1 + \epsilon^{-2La} + \epsilon^{-4La} + \epsilon^{-6La} + \,.\,.\,.\,.\,.\quad \text{``} \tag{47}$$

$$= \frac{2\,I_0\,\epsilon^{-La}}{1 - \epsilon^{-2La}} = \frac{I_0\,2}{\epsilon^{La} - \epsilon^{-La}} = \frac{I_0}{\sinh La} = \frac{E}{z_0 \sinh La} \quad \text{``} \tag{48}$$

The result (48) thus agrees with I_B in (21) or the final receiving-end-impedance of the grounded circuit is $z_0 \sinh La$ ohms.

Similarly, the final current strength at A will be the vector sum of the initial outgoing current plus all the subsequently added reflected currents, or

$$I_A = I_0 + 2\,I_0\,\epsilon^{-2La} + 2\,I_0\,\epsilon^{-4La} + 2\,I_0\,\epsilon^{-6La} + \,.\,.\,.\,.\quad \text{amperes} \tag{49}$$

$$= I_0\{1 + 2\,\epsilon^{-2La}\,(1 + \epsilon^{-2La} + \epsilon^{-4La} + \,.\,.\,.\,.\quad \text{``} \tag{50}$$

$$= I_0\left\{1 + \frac{2\,\epsilon^{-2La}}{1 - \epsilon^{-2La}}\right\} = I_0\left(\frac{1 + \epsilon^{-2La}}{1 - \epsilon^{-2La}}\right) = I_0 \coth La \quad \text{``} \tag{51}$$

$$= \frac{E}{z_0 \tanh La} \quad \text{``} \tag{52}$$

The result in (52) thus agrees with the value for I_A in (21).

If now the line be grounded through an impedance z_r ohms at B, the current $I_0 \epsilon^{-La}$ arriving at B after the first passage, is partly reflected with inversion as from a discontinuity, and the rest of it passes through the impedance z_r to ground without further reflection. Let m be the fraction of the current that goes through z_r. This coëfficient m may be called the transmission-coefficient of the impedance z_r. We have

$$m = \frac{2\, z_0}{z_0 + z_r} \tag{53}$$

a vector. Consequently the remaining portion of current $1 - m$ is reflected with inversion, as from a discontinuity, and may be called the reflection-coefficient of the impedance z_r.

$$1 - m = \frac{z_r - z_0}{z_0 + z_r} \tag{54}$$

This portion of current retreats with its sign reversed.

The current which flows through the receiving impedance z_r on first passage is $m\ I_0\ \epsilon^{-La}$ amperes, and a current $-\ (1\ -\ m)\ I_0\ \epsilon^{-La} = (m\ -\ 1)\ I_0\ \epsilon^{-La}$ returns to the sending end. It reaches that end in the condition $(m\ -\ 1)\ I_0\ \epsilon^{-2La}$ and is reflected from the ground back to B without inversion. On reaching B the second time, it will have dwindled to $(m\ -\ 1)\ I_0\ \epsilon^{-3La}$ amperes and subdivides. The portion $m\ (m\ -\ 1)\ I_0\ \epsilon^{-3La}$ passes directly to ground through z_r, and the remainder is reflected homewards as $(m\ -\ 1)^2\ I_0\ \epsilon^{-3La}$. The final current strength built up in the receiving impedance z_r will be :

$$I_B = m\, I_0\, \epsilon^{-La} + m\, (m - 1)\, I_0\, \epsilon^{-3La} + m\, (m - 1)^2\, I_0\, \epsilon^{-5La} + \cdots \tag{55}$$

$$= mI_0\epsilon^{-La}\{1 + (m - 1)\epsilon^{-2La} + (m - 1)^2\epsilon^{-4La} + (m - 1)^3\epsilon^{-6La} + \cdots$$

$$= \frac{mI_0\, \epsilon^{-La}}{1 - (m - 1)\, \epsilon^{-2La}} = \frac{mI_0}{\epsilon^{La} - (m - 1)\, \epsilon^{-La}} = \frac{I_0}{\dfrac{\epsilon^{La}}{m} - \left(\dfrac{m - 1}{m}\right)\epsilon^{-La}}$$

$$= \frac{I_0}{\left(\dfrac{2 - m}{m}\right)\cosh La + \sinh La} = \frac{I_0}{\dfrac{z_r}{z_0}\cosh La + \sinh La} \quad \text{amperes}$$

$$= \frac{E}{z_0 \sinh La + z_r \cosh La} \tag{56}$$

The final result of (56) is thus identical with the value of I_B in (23). The receiving-end-impedance with the impedance z_r at B is

$$Z_l = z_0 \sinh La + z_r \cosh La \qquad \text{ohms} \tag{57}$$

Following the above line of reasoning, we can readily understand that every discontinuity in a circuit, such as a leak or a lump of impedance at a point, sets up a pair of secondary waves; namely, those that go on through the impedance, and those that go back reflected. The final steady state is the harmonic variation of a vector sum at each point, due to all the primary, secondary, tertiary, etc., waves, summed to infinity. In many cases this vector sum may be very simply expressed in hyperbolic functions; but if these functions are not made use of, there is a fearful and wonderful summation.